The HKR filtration on HH

Recall: k comm ring ($\cdot \mathbb{Z}_{(p)}$). A animated k-alg, $X = \mathrm{Spec}(A)$

i)

$$HH(A/k) = A \underset{A \otimes_k A}{\otimes} A \simeq R\Gamma(LX, \mathcal{O}),$$

where $LX = \underline{\mathrm{Map}}(S^1, X) \in dSt_k = \mathrm{PShv}_{\mathrm{Spaces}}(\mathrm{CAlg}_k^{an})$

Note: If A is ordinary, then $LX|_{\mathrm{CAlg}_k^{disc}} \cong X$:

$$LX(R) = \mathrm{Map}(S^1 \times \mathrm{Spec}(R), X)$$

sheaf of sets on ordinary rings

$\therefore$ all maps $S^1 \times \mathrm{Spec}(R) \longrightarrow X$ factor over $\mathrm{Spec}(R)$

(2) We lifted $S' \in dSt_k$ to $S'_{Fil} \longrightarrow \mathbb{A}^1/\mathbb{G}_m \in dSt_k$

via $S'_{Fil} = BH/\mathbb{G}_m$, where $H = \mathrm{Ker}\left(F \cdot [\lambda^{p-1}] : W \times \mathbb{A}^1 \to W \times \mathbb{A}^1\right)$

- $(S'_{Fil})^u = B\mathrm{Fix} \cong \mathrm{Aff}(S')$, where $\mathrm{Fix} = \ker(F - 1)$

- $(S'_{Fil})^{gr} = \underline{B\mathrm{Ker}}/\mathbb{G}_m$, where $\mathrm{Ker} = \ker(F)$

<u>Goal</u>: Use (2) to endow $HH(A/k)$ with filtration with $gr^* \cong \mathrm{Sym}^*_A\left(L_{A/k}[1]\right)$

Def: $L_{Fil} X = \underline{\mathrm{Map}}_{\mathbb{A}^1/\mathbb{G}_m}(S^1_{Fil}, X \times \mathbb{A}^1/\mathbb{G}_m) \simeq \underline{\mathrm{Map}}(S^1_{Fil}, X)$

$\in dSt_{/(\mathbb{A}^1/\mathbb{G}_m)}$ $\quad\circlearrowleft S^1_{Fil}$ action

Lemma: $L_{Fil} X$ is an affine derived scheme over $\mathbb{A}^1/\mathbb{G}_m$

Pf idea: (1) $L_{Fil} X \big|_{\substack{\text{ordinary} \\ \text{rings}}} = X$

(2) Deduce the rest from Lurie's rep theorem

Key pt: $L_{Fil} X$ has a well-behaved cotangent complex (5.1.10 in Halpern-Leistner - Preygel)

Rmk: $$(\mathcal{L}_{\mathbb{F}_1} X)^{u} = \underline{\mathrm{Map}}(B\mathrm{Fix}, X)$$

$$(\mathcal{L}_{\mathbb{F}_1} X)^{gr} = \underline{\mathrm{Map}}(B\mathrm{Ker}, X) \circlearrowleft \mathbb{G}_m$$

Thm: k, A, X as before

1) The map $S^1 \longrightarrow \mathrm{Aff}(S^1) = B\mathrm{Fix}$ induces

$$\begin{array}{ccc} \underline{\mathrm{Map}}(B\mathrm{Fix}, X) & \xrightarrow{\sim} & \underline{\mathrm{Map}}(S^1, X) \\ \| & & \| \\ (\mathcal{L}_{\mathbb{F}_1} X)^{u} & & \mathcal{L}X \end{array}$$

2) There is a natural equiv

total space of shifted tgt complex

$$(L_{fil} X)^{gr} = \underline{\mathrm{Spec}}_X \left(\mathrm{Sym}^* \left(L_{X/k}[1] \right) \right) =: T_X[-1]$$

<u>Cor</u>: $\exists$ a natural filt on $HH(A/k)$ with $gr^* \simeq \mathrm{Sym}^*(L_{A/k}[1])$

<u>Pf</u>: Apply Thm + Lemma + base change

Pf of $\underline{\mathrm{Map}}(BFix, X) \simeq LX$:

- check on the functor of points
- reduce to $X = \mathbb{A}^1 = \mathbb{G}_a$ (use compat. with limits)

$\therefore$ Goal : $\underline{\mathrm{Map}}(BFix, \mathbb{G}_a)(R) \xrightarrow{\sim} \underline{\mathrm{Map}}(S^1, \mathbb{G}_a)(R) \quad \forall\ R \in \mathrm{CAlg}_k^{\mathrm{cn}}$

$$\begin{aligned} \mathrm{LHS} &= \mathrm{Map}(\mathrm{Spec}(R) \times BFix, \mathbb{G}_a) \\ &= \tau^{\leq 0}(R\Gamma(BFix \times \mathrm{Spec}(R)), \mathbb{G}_a) \\ &= \tau^{\leq 0}(R\Gamma(BFix, \mathcal{O}) \otimes_k R) \end{aligned}$$

$$\mathrm{RHS} = \underline{\mathrm{Map}}(S^1, \mathbb{G}_a)(R)$$

$$= \mathrm{Map}(\mathrm{Spec}(R) \times S^1, \mathbb{G}_a)$$

$$= \tau^{\leq 0}\, R\Gamma(S^1 \times \mathrm{Spec}(R), \mathbb{G}_a)$$

$$= \tau^{\leq 0}\left(R\Gamma(S^1, k) \otimes_k R\right)$$

$\therefore$ Claim follows from $R\Gamma(B\mathrm{Fix}, \mathcal{O}) \xrightarrow{\sim} R\Gamma(S^1, k)$

b/c $B\mathrm{Fix} = \mathrm{Aff}(S^1)$

Pf of $\underline{\mathrm{Map}}(B\mathrm{Ker}, X) \cong \underline{\mathrm{Spec}}_X\left(\mathrm{Sym}^*\left(L_{X/k}[1]\right)\right)$

<u>Strategy</u>: (A) Build a map $\longrightarrow$, (B) check it's an isom.

(A): Have a canonical comm square

$$\begin{array}{ccc} \mathrm{Spec}\left(k[\varepsilon]/\varepsilon^2\right) & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & B\mathrm{Ker} \end{array}$$

determined by $[\varepsilon]$:

$$\text{Square} \Leftrightarrow \text{pt of } \Omega\,\mathrm{Map}\left(\mathrm{Spec}\left(k[\varepsilon]/\varepsilon^2\right), B\mathrm{Ker}\right)$$

$$\mathrm{Ker}\left(k[\varepsilon]/\varepsilon^2\right) = \mathrm{Ker}\left(F : W(k[\varepsilon]) \longrightarrow W(k[\varepsilon])\right) \ni [\varepsilon]$$

Apply $\underline{\mathrm{Map}}(-, X)$ to this square to get:

$$\begin{tikzcd} \underline{\mathrm{Map}}(\mathrm{Bker}, X) \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & T_X = \underline{\mathrm{Spec}}_X(\mathrm{Sym}^*(L_{X/k})) \end{tikzcd}$$

dual numbers interpretation of tgt vectors

$\rightarrow$ get a map $\underline{\mathrm{Map}}(\mathrm{Bker}, X) \to X \underset{T_X}{\times} X$

$$X \underset{T_X}{\times} X \;\; // \;\; \underline{\mathrm{Spec}}_X(\mathrm{Sym}^*(L_{X/k}[1])) \;\; // \;\; T_X[-1]$$

B: this map gives an equiv

$$\underline{\mathrm{Map}}(B\ker, X)(R) \simeq T_X[-1](R) \quad \forall R \in \mathrm{CAlg}^{\mathrm{cn}}_{/k}$$

- reduce to $X = \mathbb{A}^1 = \mathbb{G}_a$

$$\mathrm{LHS} = \mathrm{Map}(B\ker \times \mathrm{Spec}(R), \mathbb{G}_a)$$

$$= \tau^{\leq 0} R\Gamma(B\ker \times \mathrm{Spec}(R), \mathcal{O})$$

$$= \tau^{\leq 0}(R\Gamma(B\ker, \mathcal{O}) \otimes_k R)$$

$$= \tau^{\leq 0}(R \oplus R[-1])$$

$$\mathrm{RHS} = \underline{\mathrm{Spec}}_{\mathbb{G}_a}(\mathrm{Sym}^*(L_{\mathbb{G}_a/k}[1]))(R) = \mathrm{Spec}(k[\underset{\deg 0}{x}, \underset{\deg 1}{\delta}])(R)$$

$$= \tau^{\leq 0}(R \oplus R[-1])$$

Check: get "obvious" map